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YIELD CURVES WITH JUMP SHORT RATES

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Abstract

This paper develops a new approach to yield curve modelling, solving analytically for bond prices when the short interest rate is a pure jump process with a rate of jump proportional to the square of an Ornstein-Uhlenbeck process. Our approach is best-suited to the problem of pricing short-dated bonds in markets in which the monetary authorities peg the short-rate, adjusting it periodically by discrete amounts.

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1 Introduction

1.1 Diffusion Models

Standard models of the yield curve derive bond prices, yields to maturity etc., under the assumption that the short interest rate is a diffusion process driven by one or more state variables. Examples are the Vasicek model in which the short interest rate is an Ornstein-Uhlenbeck process or the Cox-ingersoll-Ross model in which the short rate follows a square root process.[1]

Dissatisfaction with the empirical performance of these models has recently encouraged research on more general diffusion process models in which short rates are driven by multiple state variables. Studies of such models include Longstaff and Schwartz (1992), and Chen and Scott (1993). An empirical study based on a two-factor version of the Cox-Ingersoll-Ross model by Pearson and Sun (1994) suggests the model is still easy to reject even with the additional factor, however.

Clear from the empirical literature is the difficulty with which standard diffusion models fit the time series properties of short interest rates. In estimating square-root processes for short rates, Gibbons and Ramaswamy (1993) and Pearson and Sun (1994) find implausibly large rates of reversion to the unconditional mean suggesting serious misspecification of the stochastic processes.[2]

1.2 Interest Rate Pegging

In the present paper, we take a rather different approach to modelling the yield curve. We start from the observation that, in many bond markets, the short rate is a policy variable, pegged by the authorities and periodically adjusted in a discrete jump. Even

[1]See Vasicek (1977) and Cox, Ingersoll, and Ross (1985). Chan, Karolyi, Longstaff, and Saunders (1992) provide a useful summary and empirical investigation of nine different one-state variable models of the short interest rate. In all the models they consider (including the Vasicek and Cox-Ingersoll-Ross models), the short interest rate is assumed to be a special case of the process: $dr_t = \zeta_1(\zeta_2 - r_t)dt + \zeta_3 r_t^{\zeta_4} dW_t$ where ζ_i $i = 1, 2, 3, 4$ are constant parameters and W_t is a Brownian motion.

[2]Ball and Torous (1995) argue that the high estimated reversion rates reflect biases caused by unit roots in short rate processes. In fact, the Monte Carlos they perform suggest that only a fraction of the apparent bias can be attributed to possible unit roots.

when the short rate changes continuously over time (like the US Fed funds rate), the authorities may have a target for the market rate which periodically jumps. Furthermore, central banks typically move interest rates through a sequence of jumps, all in the same direction. Such interest rate policies have important implications for the stochastic behaviour of interest rates, particularly at the short end of the yield curve. First, the presence of jumps makes changes in short market rates highly leptokurtic. Second, the autocorrelated nature of jump sizes means the higher conditional moments including skewness and kurtosis will move around in a predictable fashion.

These observations (which will be further documented below) serve to motivate the theoretical work reported in this paper. We obtain the conditional density of short rates and the value of pure discount bonds under the assumption that short rates follow a pure jump process of which the rate of jump is a function of an Ornstein-Uhlenbeck process. In a recent paper, Babbs and Webber (1994) advocate modelling the yield curve in a similar way but ours is the first study to supply analytical solutions for interest rate densities and bond prices under such an assumption.[3]

1.3 Other Theoretical Research on Jump Risk

The more common way of introducing jumps into asset pricing models is to suppose that financial variables such as interest rates or asset values are mixed jump-diffusion processes. Typically, the jump components are taken to be standard Poisson processes with constant jump rates. Merton (1976) provided the seminal contribution on option pricing theory with jump risk while recent work includes Naik and Lee (1990), Bates (1991), Ahn (1992) and Amin (1993).

There are rather fewer papers that consider the impact of jumps on yield curve modelling from a theoretical perspective. Ahn and Thompson (1988) extend the general equilibrium pricing model of Cox, Ingersoll, and Ross (1985) to include jumps and derive pricing formulae for a simple special case. Shirakawa (1991) extends the Heath-Jarrow-Morton arbitrage yield curve model by incorporating jumps in forward

[3]Babbs and Webber (1994) discuss regularity conditions sufficient to ensure the existence of a pricing kernel in an economy without arbitrage possibilities. The pricing exercises they perform employ Monte Carlo methods to obtain numerical solutions.

interest rates. We should stress that although similar in that they include jump components in interest rates, these studies take a very different approach from ours and that of Babbs and Webber (1994) under which the rate of jump is assumed to move around stochastically, driven by an exogenous forcing process.

1.4 Structure of the Paper

Section 2 of the paper describes interest rate data from Germany, Sweden and the United States. We examine features of interest rate distributions and relate them to the monetary authorities' use of short interest rate targets. Section 3 describes our pure jump process model of short rates and demonstrates how one may derive the path density for such processes. Our technique, which draws on techniques developed in the physics literature, involves expanding the sample paths of the underlying state variable (in our case, an Ornstein-Uhlenbeck process), as an infinite sum of sine functions, using the Karhunen-Loeve Theorem. Given this expansion, we can evaluate the path density of the jump process as an expectation of a functional of the time path of the state variable. Section 4 of the paper prices bonds using the path densities. We show that bond prices are integrals of sine functions weighted by the moments of a sequence of normally distributed random variables. We also introduce risk aversion and discuss how to allow for random jump sizes.

2 Interest Rates and Monetary Policy

2.1 Key Rates in Three Markets

In this section, we document the features of short interest rate distributions that guided us in formulating our analytical yield curve model. In Figures 1 and 2, we provide data on interest rates from three different money markets, those of Germany, Sweden and the United States. In each case, the authorities control a key short term interest rate, thereby aiming to influence the market as a whole.

The key interest rates in Germany and Sweden are 'repo' rates, i.e., rates at which the authorities are willing to enter short-term repurchase agreements for longdated government securities. As one may see from Figure 1a, both key rates are pure jump processes, varying only through sharp, discontinuous movements. Further note that (i) the inter-jump times are highly variable, (ii) the signs and magnitudes of interest rate changes are highly autocorrelated, and (iii) the volatility of rates appears predictable.

Figure 1a also shows the key US interest rate, the rate on Fed funds. Clearly market-determined, the Fed funds rate nevertheless appears from the plot to move in a narrow band around a stable, underlying level which changes periodically. Such behaviour is consistent with the US authorities' publicly-acknowledged policy of intervening to keep Fed funds in a band around an implicit target level.[4]

Figures 1b and 1c show the behaviour of German and Swedish 3- and 6-month deposit rates compared to that of the respective Repo rates. One may chart the market's degree of success in forecasting Repo rate changes as evidenced by prior changes in deposit rates. For most of 1994, the Swedish market over-reacted, anticipating larger interest rate increases than actually occurred. In 1993, in both Germany and Sweden, there were sharp cuts in interest rates that appear to have been completely unanticipated by market participants. A natural way to think of what is going on here is that the market is guessing the rate of jump of the key rate controlled by the authorities.

2.2 Interest Rate Distributions

How much does the distribution of interest rate changes reflect a particular country's monetary control arrangements. One might expect that a policy of periodically adjusting a pegged short-term rate would generate quite different stochastic behaviour from a policy of controlling the money supply and leaving the interest rate to market forces. Figure 2a shows estimated densities for daily changes in German and Swedish 3-month interest rates. The densities depicted are based on non-parametric kernel estimates.[5] For each series, we standardize the data, demeaning and scaling it by the

[4]For example, for most of 1993, the official target level was 3%.

[5]The estimates are calculated using a Gaussian kernel and a window size of 1.06 $\times$ standard deviation/(sample size/5). For details, see Silverman (1986).

sample standard deviation. This enables us to compare the densities with a standard normal density also shown in Figure 2a. Evident from the plotted densities is the unconditional leptokurtosis of the interest rate changes. The Swedish daily interest rates changes are more fat-tailed than those of Germany, in that their respective kurtosis coefficients are 60.8 and 44.9.[6] It is also interesting to note that the German rate density is tri-modal, reflecting the influence on the distribution of large jumps in rates.

Figure 2b shows kernel estimates for the densities of US interest rate changes in the periods (i) January 1980 to September 1983, and (ii) October 1983 to April 1995. In the earlier period, the Federal Reserve was following a policy of targeting Non-Borrowed Reserves, effectively a measure of base money, while allowing the market to determine the Fed funds rate.[7] In the later period, the Federal Reserve targeted Borrowed Reserves. Since, these latter are directly proportional to the Fed funds rate, this policy effectively meant targeting interest rates.

The density for the earlier period corresponds reasonably closely to that of the normal distribution also shown in the figure. It is interesting to note that the kurtosis for the early period was just 6.3. While this probably represents a statistically significant deviation from the kurtosis of a normally distributed random variable, it is far less than the sample kurtosis of 13.2 that we calculate from the data after October 1983. Finally, the density for US data from the later period exhibits the same kind of tri-modal configuration commented on in the case of Germany above. Once again, sharp and comparatively large interest rates adjustments seem to account for this.

[6]Recall that the kurtosis coefficient, defined as the ratio of the fourth central moment to the square of the second, is 3 for a normally distributed random variable. The sample periods for the kurtosis calculations are 20/11/92 to 23/8/96 for the Swedish rate changes (the former date being subsequent to the floating of the Swedish Krone) and 1/1/80 to 23/8/96 for German rates.

[7]Even in this period, it should be noted, the US authorities adjusted their Non-Borrowed Reserve target in part to take account of interest rate developments, so interest rates remained to some extent a target variable.

3 The Model

3.1 Short Rates and Their Time Path Densities

Informed by our examination of the nature of interest rate distributions, we now develop a new theoretical approach to modelling short rates. Let r_t be the instantaneous interest rate. Assume r_t is a doubly stochastic Poisson process with rate of jump:

$$\gamma(X_t) = \beta X_t^2, \tag{1}$$

where X_t is a diffusion process, β is a constant, and where the jumps are of known size δ_n for jumps $n = 1, 2, \ldots, \infty$. (We shall relax the assumption that the jump sizes are known below.) Let X_t be an Orstein-Uhlenbeck process:

$$dX_t = \alpha\,(\theta - X_t)\,dt + \sigma dW_t, \tag{2}$$

where W_t is a standard Brownian motion, α, σ, and θ are constant parameters, and α, $\sigma > 0$.

We wish to derive bond prices under these assumptions. If agents are risk neutral, the price at time t of a pure discount bond yielding \$1 at T can be written:

$$P_{t,T} = E_t\left[\exp\left(-\int_t^T r_\tau d\tau\right)\right]. \tag{3}$$

To evaluate such expectations, we need to derive the probability density, conditional on information at t, of the path of r_τ up to a future date, T. We normalize our choice of time units so that $t = 0$ and $T = 1$.

Conditional on the time path followed by the forcing process, X_t, r_t is a Poisson process. Hence, (see Snyder and Miller (1991), page 358), the sample or time path density may be expressed as:

$$\rho\,[N(t) : 0 \leq \tau < 1] = E_0\left[\exp\left(-\int_0^1 \gamma(\tau, X(\tau))d\tau + \int_0^1 \ln\gamma(\tau, X(\tau))dN(\tau)\right)\right]. \tag{4}$$

In this context, the sample path density may be thought of as the joint density of the number of jumps and the times that they occur. Most of the work in this paper consists of evaluating equation (4) and hence obtaining the path density. Knowing this density permits us to evaluate not just bond prices but also the values of many other bond and interest rate derivatives.

3.2 A Series Representation for the State Variable

To solve (4), we apply an approach developed by Macchi (1971) for modelling the emission of light photons.[8] The first step is to express the Ornstein-Uhlenbeck driving process, X_t, as an infinite weighted sum of orthogonal functions of time. The orthogonality property of the representation will facilitate solution of equation (4). We state the representation, which is an application of the Karhunen-Loeve Theorem, in the form of a proposition. Proofs for this and subsequent results are provided in the Appendix.

Proposition 1 *By the Karhunen-Loeve Theorem, X_t can be written as:*

$$X_t = \sum_{n=1}^{\infty} x_n \phi_n(t) \qquad \text{for} \qquad t \in [0,1], \tag{5}$$

where the $\phi_n(t)$ for $t \in [0,1]$ are functions of the form:

$$\phi_n(\tau) = \frac{2\sin(\omega_n \tau)}{\sqrt{2 - \sin(2\omega_n)/\omega_n}}. \tag{6}$$

Here, the $\omega_1, \omega_2, \omega_3 \ldots$ are the positive roots of the equation,

$$\frac{2\alpha i \omega_n}{\omega_n^2 + \alpha^2} = \alpha \left(\frac{\exp[-\alpha + i\omega_n] - 1}{-\alpha + i\omega_n} - \frac{\exp[-\alpha - i\omega_n] - 1}{-\alpha - i\omega_n} \right). \tag{7}$$

The x_n are real-valued, normally distributed independent random variables with variances, $v_n^2 = \sigma^2/(\omega_n^2 + \alpha^2)$ while the means, $m_n \equiv \mathrm{E}_0 x_n$, are given by:

$$m_n = \frac{\eta_n (X_0 - \theta)}{\omega_n^2 + \alpha^2} \{\omega_n - \exp[-\alpha](\alpha \sin(\omega_n) + \omega_n \cos(\omega_n))\} + \frac{\eta_n \theta}{\omega_n}(1 - \cos(\omega_n)), \tag{8}$$

where $\eta_n \equiv 2/\sqrt{2 - \sin(2\omega_n)/\omega_n}$.

[8] Discussion of these techniques may be found in Daley and Vere-Jones (1988) and Snyder and Miller (1991). The physics applications are generally concerned with the *unconditional* representation of Gaussian processes in some time interval $[0,1]$. In other words, the stochastic behaviour of X_t for $t \in [0,1]$ is studied assuming that X_0 is a draw from the unconditional distribution of the level of the process. This leads to a different eigenfunction expansion from that given in Proposition 1 since the boundary conditions satisfied by the eigenfunctions differ from those that apply if the analysis is conducted conditional on X_0.

An important issue here is the magnitude of the errors introduced if one truncates the infinite sum in equation (5) at some finite number, N. Any practical exercise using the Karhunen-Loeve expansion will, of course, require such a truncation. To approximate the actual sample paths of the Ornstein-Uhlenbeck process accurately would require a large number of terms in the expansion as the process sample paths are of infinite variation while the eigenfunctions are smooth. However, since we wish to employ the expansion ultimately to calculate an expectation of a functional of the future time path of the process (i.e., the functional appearing in equation (4)), more important is the degree to which we can approximate the conditional moments of X_t.

Figure 3 shows the conditional mean and variance of X_t calculated using the exact formulae for these moments (denoted 'true value' in the figure) and approximations based on a truncated version of the infinite sum in equation (5). The truncations range from a single ϕ_n function to 100 terms. The calculation is carried out for an Ornstein-Uhlenbeck process with parameters, $\beta = 8$, $\alpha = 0.4$, $\theta = 0.8$, $X_0 = 0.6$, $\delta = -0.005$, and $\sigma = 0.3$. As one may see, the main source of potential problems is the mean approximation. The eigenfunctions are sine functions and hence all equal zero at $t = 0$, whereas the true conditional mean,

$$m_{0,t} \equiv \mathrm{E}_0(X_t) = \theta + \exp[-\alpha t](X_0 - \theta) \qquad \text{for} \quad t \in [0, t], \tag{9}$$

is strictly positive at $t = 0$. As one adds eigenfunctions, the approximation to $m_{0,t}$ improves in a mean square sense, but convergence is relatively slow. The next proposition gives a simple modification of the series expansion that perfectly matches the conditional mean for all truncations and which fits the conditional variance just as well as the expansion in Proposition 1.

Proposition 2 *Consider the process, $X_t^{(N)}$, defined by:*

$$X_t^{(N)} = x_0 \phi_0(t) \; + \; \sum_{n=1}^{N} x_n \phi_n(t) \qquad \text{for} \qquad t \in [0, 1], \tag{10}$$

where x_n and $\phi_n(t)$ for $n = 1, 2, \ldots$ are defined as in Proposition 1, and where:

$$\phi_0(t) \equiv \frac{\psi(t)}{\sqrt{\int_0^1 \psi(s)^2 ds}} \tag{11}$$

$$x_0 \equiv \sqrt{\int_0^1 \psi(s)^2 ds} \quad (12)$$

$$\psi(t) \equiv m_{0,t} - \sum_{n=1}^{N} \left[\left(\int_0^1 m_{0,s}\phi_n(s)ds \right) \phi_n(t) \right]. \quad (13)$$

For all $N \geq 1$ and $t \in [0,1]$,

$$\mathrm{E}_0\left(X_t^{(N)}\right) = m_{0,t} \quad (14)$$

$$\mathrm{Var}_0\left(X_t^{(N)}\right) = \mathrm{Var}_0\left(\sum_{n=1}^{N} x_n\phi_n(t)\right), \quad (15)$$

i.e., the conditional mean of $X_t^{(N)}$ exactly matches that of X_t, while the conditional variance of $X_t^{(N)}$ equals that of the expansion appearing in Proposition 1, truncated at N eigenfunctions.

Adding the extra eigenfunction does not affect the conditional variance since it is multiplied by a deterministic quantity, x_0, however, it dramatically improves the approximation to the mean by adding the mean function less its projection on the N first eigenfunctions. Using the modified projection of Proposition 2, the variance approximations are as they appear in Figure 3b while the mean is perfectly fitted. As a Gaussian process, the conditional distribution of X_t at $t = 0$ is completely characterised by its conditional mean and variance. Hence, we may conclude that the higher moments of the conditional distributions are well fitted.

3.3 The Time Path Density for Interest Rates

The orthogonality properties of the Karhunen-Loeve representation enable one to write the expectation in equation (4) in the following simple form:

$$\rho\left[N(\tau) : 0 \leq \tau < 1\right] = E_0 \left[\exp\left[-\beta \sum_{s=1}^{\infty} x_s^2 + \int_0^1 \ln \beta X(\tau)^2 dN(\tau) \right] \right], \quad (16)$$

where $N(\tau)$ is the counting process associated with the Poisson process. Evaluating equation (16), we obtain the following result.

Proposition 3 *If the jump times are denoted t_1, t_2, t_3, ...,*

$$\rho\left[N(\tau):0\leq\tau<1\right] = a_0+\sum_{L=1}^{\infty}\beta^L \sum_{\substack{j_1,\ldots j_L=1\\ k_1,\ldots k_L=1}}^{\infty} a_L(j,k)\prod_{l=1}^{L}\phi_{j_l}(t_l)\phi_{k_l}(t_l), \quad (17)$$

where:

$$a_0 = E_0\left[\exp\left[-\beta\sum_{n=1}^{\infty}x_n^2\right]\right] \quad (18)$$

$$a_L(j,k) = \prod_{n=1}^{\infty}E_0\left[x_n^{p_n}\exp\left[-\beta x_n^2\right]\right], \quad (19)$$

where $j\equiv(j_1,\ldots j_L)$ and $k\equiv(k_1,\ldots k_L)$ are L-dimensional permutations of the positive integers and where $p_n=p_n(j,k)$ is the total number of elements in the two permutations that equal n. The evaluation of the $a_L(j,k)$ is somewhat involved and is described in detail in the Appendix.

Proposition 3 gives the path density for the jump process followed by the short interest rate. Since the jump sizes are assumed to be known, the sample path density may be thought of as the joint density of the number of jumps and the individual jump times. Integrating over the jump times yields the probabilities associated with different numbers of jumps between times 0 and 1. Though complicated, the density is reasonably straightforward to compute. To verify our analytical solution, we simulated the jump process and estimated the probabilities of different numbers of jumps by Monte Carlo. Employing 200,000 replications with 500 time steps between 0 and 1 and reducing the variance of the Monte Carlo estimates using antithetic variate techniques, we found the probablities were extremely close to those implied by our analytic path densities. Software to perform these calculations is available from the authors on request.

4 Bond Prices

4.1 Bond Pricing with Known Jump Size

In this section, we describe how one may calculate bond prices using the path density derived in Proposition 3. Recall that this involves evaluating the expectation in equation (3). For the moment, we shall maintain our assumption that the jump sizes, δ_j, $j = 1, 2, \ldots, \infty$, are known constants. At any time, t, the level of interest rates, r_t, may be written as:

$$r_t = r_0 + \sum_{j=1}^{N(t)} \delta_j, \tag{20}$$

where $N(t)$ is the number of jumps up to and including time t. The integral in equation (3) for $t = 0$ and $T = 1$, can then be written:

$$\int_0^1 r_s ds = r_0 t_1 + (r_0 + \delta_1)(t_2 - t_1) + (r_0 + \delta_1 + \delta_2)(t_3 - t_2) + \ldots + \left(r_0 + \sum_{j=1}^{N(1)} \delta_j \right) (1 - t_{N(1)}). \tag{21}$$

Cancelling terms, one may then write the bond price as:

$$P_{0,1} = E_0 \left[\exp \left(-r_0 - \sum_{j=1}^{N(1)} \delta_j (1 - t_j) \right) \right]. \tag{22}$$

Evaluating this expectation, we have:

Proposition 4 *For known jumps sizes, δ_j, $j = 1, 2, \ldots \infty$, the price at $t = 0$ of a pure discount bond maturing at time $t = 1$ is:*

$$P_{0,1} = a_0 \exp[-r_0] + \exp[-r_0] \sum_{L=1}^{\infty} \beta^L \sum_{\substack{j_1, \ldots j_L = 1 \\ k_1, \ldots k_L = 1}}^{\infty} a(j,k) \int_0^1 \tilde{\phi}_{j_1,1}(\tau_1) \tag{23}$$

$$\times \tilde{\phi}_{k_1,1}(\tau_1) \left[\int_{\tau_1}^1 \tilde{\phi}_{j_2,2}(\tau_2) \tilde{\phi}_{k_2,2}(\tau_2) \left[\ldots \left[\int_{\tau_{L-1}}^1 \tilde{\phi}_{j_L,L}(\tau_L) \tilde{\phi}_{k_L,L}(\tau_L) d\tau_L \right] \ldots \right] d\tau_2 \right] d\tau_1, \tag{24}$$

where $\tilde{\phi}_{j,n}(\tau) \equiv \exp[-\delta_n(1 - \tau)/2] \phi_j(\tau)$.

The expression for the bond price may appear complicated and the number of terms in the infinite sums computationally costly to calculate. However, an important feature of the Karnunen-Loeve expansion is that, as we noted in Section (3.2), the rate of convergence of the series expansion is very rapid. Hence, it is necessary to calculate only a very small fraction of the various permutations of positive integers in equation (23) in order to reach reasonable levels of accuracy.

The algorithm we developed to calculate bond prices is described in the Appendix. We checked the accuracy of the algorithm using Monte Carlos. For typical short bond valuations, the Monte Carlos required several minutes of CPU time on a 100 Mhz Pentium computer to achieve acceptable accuracy. Evaluation of our analytic expressions was much quicker and rarely exceeded a minute. Software to perform such calculations is available on request from the authors.

Figure 4 shows yield curves for short bonds implied by our model, calculated for different value of the initial level of the state variable, X_0. The jump sizes assumed are -1/2% so the yield curves are downward sloping. The rate of reversion, α, of the X_t process to its unconditional mean, θ, is relatively slow so X_0 has a sizeable inpact on the expected number of jumps over the period 0 to 1. Figure 5 shows yield curves for a single value of X_0 but for different levels of σ. As σ becomes small, bond yields approach those that would apply if the short rate were a Poisson process, i.e., with known but time-varying jump rate.

4.2 Risk Aversion

In this section, we sketch how one may generalise our bond pricing formula to the case in which agents are risk averse. This involves a simple application of change of measure arguments. Under weak conditions in the absense of arbitrage opportunities, Harrison and Kreps (1979) show that there exists a random variable, q, such that the price at $t = 0$, $\Pi_0(Z)$, of an asset that pays a random amount Z at time $t = 1$ may be expressed as:

$$\Pi_0(Z) = \mathrm{E}_0(qZ). \tag{25}$$

In this context, q is referred to as a pricing kernel. Under risk neutrality, $q = \exp\left[-\int_0^1 r_s ds\right]$ where r_s is the short-term interest rate. Hence, the value of a pure discount bond which pays one dollar at time 1 is $\mathrm{E}_0\left(\exp\left[-\int_0^1 r_s ds\right]\right)$, similar to the expression in equation (3). One may easily show that if $q = \exp\left[-\int_0^1 r_s ds\right]$, all assets must grow on average at a proportional rate r_t, whence the term risk neutral.

Suppose that interest rates follow the jump process hypothesised in equations (1) and (2), but that the pricing kernel q is more general. In particular, assume that the pricing kernel, q, is equal to:

$$\begin{aligned} q &= \exp\left[-\int_0^1 r_s ds\right] \exp\left[\int_0^1 (1-\kappa_{1s})\beta X_s^2 ds + \int_0^1 \log(\kappa_{1s}) dN_s\right] \\ &\times \exp\left[\int_0^1 \kappa_{2s} dW_s - \int_0^1 \kappa_{2s}^2/2ds\right]. \end{aligned} \tag{26}$$

where κ_{it} for $i = 1, 2$ are non-negative, predictable processes, N_t is the counter for the doubly stochastic Poisson process described in equations (1) and (2) and W_t is the Brownian motion that appears in equation (2). Subject to regularity conditions (see Bremaud (1981) Theorem 3, Chapter 6, and Lipster and Shiryayev (1977) Section 6.3) the price at $t = 0$ of a discount bond paying off at $t = 1$ is:

$$P_{0,1} = E_0^*\left[\exp\left(-\int_t^T r_s ds\right)\right]. \tag{27}$$

where $E_0^*(.)$ is the expectations operator associated with risk-adjusted or 'risk-neutral' probabilities. Under these risk-neutral probabilities, the short interest rate, r_t, is a point process with rate of jump $\kappa_{1t}\beta X_t^2$ and the process X_t follows the process:

$$dX_t = \alpha(\theta - X_t)dt + \kappa_{2t}\sigma dt + \sigma d\tilde{W}_t, \tag{28}$$

where $\tilde{W}_t$ is a Brownian motion. With this more general pricing kernel, asset prices will not grow at a proportional rate equal to the safe instant rate.

4.3 Bond Pricing with Stochastic Jump Sizes

An important simplifying assumption maintained up to this point is that the jump sizes are of known magnitude. Clearly, this assumption is extremely strong.[9] In this section, we discuss various ways in which it may be relaxed.

[9]It is the assumption followed by several significant papers in the literature including Ahn and Thompson (1988) in their extension of the Cox-Ingersoll-Ross model to include jumps.

One possibility would be to incorporate random jump sizes distributed independently for different jumps. In general, this has been the approach taken in the finance literature when jump components have been included in pricing models (see Merton (1976), Bates (1991), and Ho, Perraudin, and Sorensen (1996) for three examples among many).

However, even cursory inspection of the data reveals that successive jump changes in short interest rate changes are highly serially correlated. In general, central banks push interest rates in one direction over a period of time before eventually reversing the movement. In what follows, we shall capture this type of behaviour by supposing that the size of the jump is the current level of an arithmetic Brownian motion that evolves over time.

The advantages of this assumption are (i) that successive jump magnitudes are highly correlated, especially if they occur close together in time, and (ii) that jump sizes can be either positive or negative. Possible limitations of our approach include the fact that we shall assume that jump magnitudes are independent of jump times[10] and that jump sizes (and, indeed, jump times) are taken to be independent of the level of interest rates.[11]

The last and perhaps most serious limitation with this approach is that it implies that financial market participants know the size of an interest rate change should one occur in the next instant of time. In other words, the jump magnitude is instantaneously predictable. This is obviously not consistent with reality but appears to be a reasonable simplifying assumption given the other advantages of this kind of specification.

Let us state the pricing result with random jumps sizes as a proposition:

Proposition 5 *Suppose that, conditional on the jump times, $t_1, t_2, \ldots$, the jump sizes equal the contemporaneous levels of an arithmetic Brownian motion, Z_t, with drift $\tilde{\mu}$ and instantaneous standard deviation, $\tilde{\sigma}$, i.e., $\delta_j = Z_{t_j}$ for $j = 1, 2, \ldots, \infty$, where Z_t is independent of the jump realizations and of the state variable, X_t. Then, bond*

[10]In fact, this seems a reasonably innocuous assumption as there is unlikely to be much dependable stucture in the correlation of jump times and magnitudes.

[11]Babbs and Webber (1994) suggest that the interest rate level could be an important determinant of the stochastic nature of jumps.

prices are:

$$
\begin{aligned}
P_{0,1} &= a_0 \exp[-r_0] + \exp[-r_0] \sum_{L=1}^{\infty} \beta^L \sum_{\substack{j_1,\ldots j_L = 1 \\ k_1,\ldots k_L = 1}}^{\infty} a(j,k) \int_0^1 \int_{t_1}^1 \cdots \int_{t_{L-1}}^1 \\
&\times \exp\left(\sum_{m=1}^{L} \xi_m Z_0\right) \exp\left(\sum_{i=1}^{L}\left(\sum_{l=i} \xi_l\right) \tilde{\mu}(\xi_i - \xi_{i-1}) + \left(\sum_{l=i}^{L} \xi_l\right)^2 \tilde{\sigma}^2(\xi_i - \xi_{i-1})/2\right) \\
&\times \tilde{\phi}_{j_1}(\tau_1)\tilde{\phi}_{k_1}(\tau_1)\tilde{\phi}_{j_2}(\tau_2)\tilde{\phi}_{k_2}(\tau_2)\ldots\tilde{\phi}_{j_L}(\tau_L)\tilde{\phi}_{k_L}(\tau_L) d\tau_L \ldots d\tau_2 d\tau_1. \qquad (29)
\end{aligned}
$$

where $\xi_i \equiv \tau_i - 1$.

5 Conclusion

This paper has developed a new analytical framework for studying fixed income securities. Specifically, we suppose that the instantaneous interest rate is a pure jump process, periodically adjusted by monetary authorities. These assumptions are designed to capture features of the monetary policy arrangements operated by such countries as Germany, Sweden and the United Kingdom. As we argue in the Introduction, our model may also be applicable to United States interest rates, especially in periods in which the Federal Reserves has targeted particular level for the Fed funds rate.

In general, we think it is interesting to study ways in which the monetary arrangements employed in different countries affect the behaviour of their domestic interest rates. Such 'market microstructure' approaches to the study of monetary policy and the yield curve have been relatively little explored. Recent papers that have examined the broader implications of specific aspects of central bank behaviour are Balduzzi, Bertola, and Foresi (1993) and Vitale (1996).

In future work, we plan to implement our model empirically. Our framework is capable of mimiking generally recognised features of short interest rate and bond yield distributions including extreme unconditional leptokurtosis, and predictable time variation in the second and higher moments. It, therefore, seems well-suited for empirical applications.[12]

[12]For example, there has been much recent interest in using yield curve models to infer market

APPENDIX

This Appendix provides reasonably full and self-contained derivations of the results described in the text. Let r_t be the short interest rate and assume that r_t follows a pure jump process with rate of jump:

$$\gamma(X_t) = \beta X_t^2, \tag{30}$$

where X_t is a diffusion process and where the jumps are of known size δ_n for jumps $n = 1, 2, \ldots, \infty$. Let:

$$dX_t = \alpha(\theta - X_t)\, dt + \sigma dW_t, \tag{31}$$

where dW_t is a standard Brownian motion increment and α, σ, and θ are constant parameters, and α, $\sigma > 0$. We wish to derive the probability density, conditional on information at t_0, of the path of r_t up to a future date, t_1, under these assumptions. Normalize our choice of time units so that $t_0 = 0$ and $t_1 = 1$.

Conditional on the time path followed by the forcing process, X_t, r_t is a Poisson process. Hence, the sample path density conditional on information at $t = 0$, ρ, may be expressed as:

$$\rho\,[N(t) : 0 \leq \tau < 1] = E_0 \left[\exp\left(-\int_0^1 \gamma(\tau, X(\tau))d\tau + \int_0^1 \ln\gamma(\tau, X(\tau))dN_\tau\right)\right]. \tag{32}$$

Most of the derivation consists of evaluating this expectation. To calculate the expectation, it is convenient to express X_t in terms of a Karhunen-Loeve expansion.

Proof of Proposition 1

Derivation of the Time Path Density

As an Ornstein-Uhlenbeck process, X_t can be written as:

$$X_t = \theta + \exp[-\alpha t]\,(X_0 - \theta) + \sigma\int_0^t \exp[-\alpha(t-\tau)]\, dW_\tau \equiv m_{0,t} + \epsilon_{0,t}, \tag{33}$$

where $m_{0,t} \equiv \theta + \exp[-\alpha t]\,(X_0 - \theta)$. We note that:

$$K(\tau, u) = K(\tau - u) = E_0\,[\epsilon_{0,\tau}\epsilon_{0,u}] = \frac{\sigma^2}{2\alpha}\left(\exp[-\alpha|\tau - u|] - \exp[-\alpha(\tau + u)]\right). \tag{34}$$

expectations of future interest rates, inflation and monetary policy (see, for example, Dahlquist and Svensson (1994)). Implementing our model on time series of bond prices, one may invert the pricing formulae to obtain estimates of the market's view of the likelihood of interest rate changes.

By the Karhunen-Loeve Theorem, X_t can be expressed as:

$$X_t = \sum_{n=1}^{\infty} x_n \phi_n(t) \qquad \text{for} \quad t \in [0,1], \tag{35}$$

where the x_n are independent random variables distributed normally with means m_n and variance v_n. The $\phi_n(t)$ for $t \in [0,1]$ are the eigenfunctions of the covariance function for the stochastic process X_t.

Evaluating the Eigenfunctions

To determine the eigenfunctions, one must solve the integral equation:

$$\begin{aligned}
\lambda_n \phi_n(\tau) &= \int_0^1 K(\tau,u)\phi_n(u)du & (36)\\
&= \int_0^\tau K(\tau,u)\phi_n(u)du + \int_\tau^1 K(\tau,u)\phi_n(u)du & (37)\\
&= \int_0^\tau \frac{\sigma^2}{2\alpha}\left(\exp[-\alpha(\tau-u)] - \exp[-\alpha(\tau+u)]\right)\phi_n(u)du \\
&\quad + \int_\tau^1 \frac{\sigma^2}{2\alpha}\left(\exp[-\alpha(u-\tau)] - \exp[-\alpha(\tau+u)]\right)\phi_n(u)du. & (38)
\end{aligned}$$

The two terms on the right-hand-side of equation (36), come from the fact that we have an absolute value operator in the Ornstein-Uhlenbeck covariance function $K(\tau,u)$. Here, the λ_n $n = 1...\infty$ are eigenvalues yet to be determined. Taking derivatives yields:

$$\begin{aligned}
\hat{\lambda}_n \phi_n'(\tau) &= -\alpha \int_0^\tau \left(\exp[-\alpha(\tau-u)] - \exp[-\alpha(\tau+u)]\right)\phi_n(u)du \\
&\quad +\alpha \int_\tau^1 \left(\exp[-\alpha(u-\tau)] + \exp[-\alpha(\tau+u)]\right)\phi_n(u)du. & (39)
\end{aligned}$$

where $\hat{\lambda}_n \equiv \lambda_n 2\alpha/\sigma^2$. Differentiating again yields:

$$\hat{\lambda}_n \phi_n''(\tau) = \hat{\lambda}_n \alpha^2 \phi_n(\tau) - \alpha\left(1 - \exp[-2\alpha\tau]\right)\phi_n(\tau) - \alpha\left(1 + \exp[-2\alpha\tau]\right)\phi_n(\tau). \tag{40}$$

Therefore:

$$\phi_n''(\tau) + \left[\frac{2\alpha}{\hat{\lambda}_n} - \alpha^2\right]\phi_n(\tau) = 0 \tag{41}$$

The general solution to this equation is:

$$\phi_n(\tau) = A_{1,n}\exp\left[i\omega_n\tau\right] + A_{2,n}\exp\left[-i\omega_n\tau\right], \tag{42}$$

where:

$$\omega_n = \left[\frac{2\alpha}{\hat{\lambda}_n} - \alpha^2\right]^{\frac{1}{2}} \qquad \text{i.e.} \qquad \hat{\lambda}_n = \left[\frac{\omega_n^2}{2\alpha} + \frac{\alpha}{2}\right]^{-1} = \frac{2\alpha}{\omega_n^2 + \alpha^2}. \tag{43}$$

To see this, take derivatives of equation (42) and substitute into equation (41) to obtain:

$$A_{1,n}\left((i\omega)^2+\omega^2\right)\exp\left[i\omega_n\tau\right]+A_{2,n}\left((-i\omega)^2+\omega^2\right)\exp\left[-i\omega_n\tau\right]=0. \tag{44}$$

Boundary Conditions

The integral equation effectively imposes boundary conditions on the solution to the differential equation. Since $K(0,u)=K(\tau,0)=0$ for all τ, $u\geq 0$, $\phi_n(0)=0$ for $n=1,2,\ldots$. Hence, $A_{1,n}=-A_{2,n}$ for $n=1,2,\ldots$. Let $A_n\equiv A_{1,n}$. Substituting $\phi_n(\tau)=A_n(\exp[i\omega\tau]-\exp[-i\omega_n\tau])$ into equation (39), we obtain:

$$\begin{aligned}\hat{\lambda}_n\phi_n'(\tau) &= \hat{\lambda}_nA_ni\omega_n\left(\exp[i\omega_n\tau]+\exp[-i\omega\tau]\right) && (45)\\ &= -\alpha\int_0^\tau\left(\exp[-\alpha(\tau-u)]-\exp[-\alpha(\tau+u)]\right)A_n\left(\exp[i\omega_nu]-\exp[-i\omega_nu]\right)du+ \\ &\quad\ \alpha\int_\tau^1\left(\exp[-\alpha(u-\tau)]+\exp[-\alpha(\tau+u)]\right)A_n\left(\exp[i\omega_nu]-\exp[-i\omega_nu]\right)du. && (46)\end{aligned}$$

Evaluating at $\tau=0$ gives

$$\begin{aligned}2\hat{\lambda}_nA_ni\omega_n &= A_n\alpha\int_0^1\left(\exp[-\alpha u]+\exp[-\alpha u]\right)\left(\exp[i\omega_nu]-\exp[-i\omega_nu]\right)du && (47)\\ &= A_n2\alpha\left(\frac{\exp[-\alpha+i\omega_n]-1}{-\alpha+i\omega_n}-\frac{\exp[-\alpha-i\omega_n]-1}{-\alpha-i\omega_n}\right). && (48)\end{aligned}$$

Multiplying both sides by A_n^* where the asterisk indicates the complex conjugate, we obtain an implicit equation in ω:

$$\frac{2\alpha i\omega}{\omega^2+\alpha^2}=2\alpha i\Im\left\{\frac{\exp[-\alpha+i\omega]-1}{-\alpha+i\omega}\right\}, \tag{49}$$

where $\Im$ denotes the imaginary part of a complex number. One may show that equation (49) has countable roots, $\omega_1,\omega_2,\omega_3,\ldots$ which can be found numerically with little difficulty using grid search methods.

To find A_n, one may employ the normalization:

$$\begin{aligned}\int_0^1\phi_n(\tau)\phi_n^*(\tau)d\tau &= A_nA_n^*\int_0^1\left(\exp[i\omega_n\tau]-\exp[-i\omega_n\tau]\right)\left(\exp[-i\omega_n\tau]-\exp[i\omega_n\tau]\right)d\tau \\ &= A_nA_n^*\int_0^1 4\sin^2(\omega_n\tau)d\tau && (50)\\ &= A_nA_n^*\left(2-\frac{\sin(2\omega_n)}{\omega_n}\right)=1. && (51)\end{aligned}$$

Letting $A_n=-i/\sqrt{2-\sin(2\omega_n)/\omega_n}$ we finally obtain the real-valued function:

$$\phi_n(\tau)=\frac{2\sin(\omega_n\tau)}{\sqrt{2-\sin(2\omega)/\omega_n}}. \tag{52}$$

The Ornstein-Uhlenbeck process, X_t, can then be written as:

$$X_t = \sum_{n=1}^{\infty} x_n \phi_n(t) \tag{53}$$

where the x_n are independent, real-valued, normally distributed random variables satisfying:

$$x_n = \int_0^1 X_\tau \phi_n(\tau) d\tau. \tag{54}$$

Determining the x_n Means and Variances

The means of the x_n take the form:

$$\begin{aligned}
\mathrm{E}_0 x_n &= \int_0^1 \mathrm{E}_0(X_\tau)\phi_n(\tau)d\tau && (55)\\
&= \int_0^1 (\theta + (X_0 - \theta)\exp[-\alpha\tau])\,\phi_n(\tau)d\tau && (56)\\
&= \left[\frac{-\theta\gamma_n}{\omega_n}\cos(\omega s) + \gamma_n(X_0-\theta)\frac{\exp[-\alpha s]}{\alpha^2+\omega_n^2}(-\alpha\sin(\omega_n s) - \omega_n\cos(\omega_n s))\right]_0^1 && (57)\\
&= \frac{\theta\eta_n}{\omega_n}(1-\cos(\omega_n)) + \frac{\eta_n(X_0-\theta)}{\omega_n^2+\alpha^2}\left(\omega_n - \exp[-\alpha]\left(\alpha\sin(\omega_n) + \omega_n\cos(\omega_n)\right)\right) && (58)
\end{aligned}$$

where $\eta_n \equiv \frac{2}{\sqrt{2-sin(2\omega_n)/\omega_n}}$. To evaluate the variances, we simply note that:

$$\begin{aligned}
\mathrm{E}_0\left\{(X(t)-m_{0,t})(X(\tau)-m_{0,\tau}\right\} &= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\mathrm{E}_0(x_n-m_n)(x_j-m_j)\phi_n(t)\phi_j(\tau) && (59)\\
&= \sum_{n=1}^{\infty}\mathrm{Var}(x_n)\phi_n(t)\phi_n(\tau). && (60)
\end{aligned}$$

Multiplying both sides by $\phi_n(\tau)$, integrating over τ on $[0,1]$ and using the integral equation, (36) shows that $\mathrm{Var}(x_n) = \lambda_n$. This completes the proof of Proposition 1. □

Proof of Proposition 2

The idea of the additional eigenfunction, $\phi_0(t)$, is sketched in the text. The equality of the conditional means and variances is straightforward to show. □

Proof of Proposition 3

Path Densities for the Jump Process

The orthogonality of the Karhuenen-Loeve representation means that:

$$\rho\left[N(\tau):0\leq\tau<1\right]=E_0\left[\exp\left[-\beta\sum_{s=1}^{\infty}x_s^2+\int_0^1\ln\beta X(\tau)^2dN(\tau)\right]\right]. \tag{61}$$

Let $\rho_L \equiv \rho[N(\tau) : 0 \leq \tau < 1|N(1) = L] \times \text{Probo}_0\{N(1) = L\}$ for $L = 1,2,3,\ldots$ and $\rho_0 \equiv \text{Probo}_0\{N(1)=0\}$. Then, $\rho = \sum_{L=0}^{\infty} \rho_L$ is the total path density. Here,

$$\rho_0\left[N(\tau):0\leq\tau<1\right] \quad = \quad E_0\left(\prod_{n=1}^{\infty}\exp\left[-\beta x_n^2\right]\right). \tag{62}$$

For $N(1) = L \geq 1$ with jump times t_1, t_2, ..., t_L:

$$\begin{aligned}\rho_L\left[N(\tau):0\leq\tau<1\right] &= \beta^L E_0\left[X(t_1)^2X(t_2)^2...X(t_L)^2\exp\left[-\beta\sum_{s=1}^{\infty}x_s^2\right]\right]\\ &= \beta^L \sum_{\substack{j_1,...j_L=1\\k_1,...k_L=1}}^{\infty} a_L(j,k)\prod_{l=1}^{L}\phi_{j_l}(t_l)\phi_{k_l}(t_l),\end{aligned} \tag{63}$$

where:

$$a_L(j,k)\equiv E_0\left[x_{j_1}x_{j_2}...x_{j_L}x_{k_1}x_{k_2}...x_{k_L}\exp\left[-\beta\sum_{s=1}^{\infty}x_s^2\right]\right]=\prod_{n=1}^{\infty}E_0\left[x_n^{p_n}\exp\left[-\beta x_n^2\right]\right], \tag{64}$$

where $j \equiv (j_1,...j_L)$ and $k \equiv (k_1,...k_L)$ are L-dimensional permutations of the positive integers and p_n is the total number of elements in a given pair of permutaions, j and k, equal to n.

Evaluating the Higher Moments of the x_j

We wish to evaluate:

$$\begin{aligned}E_0\left[x_n^p\exp\left[-\beta x_n^2\right]\right] &= \frac{1}{\sqrt{2\pi\lambda_n}}\int_{-\infty}^{\infty}x_n^p exp\left[-\beta x_n^2\right]\exp\left[-\frac{(x_n-m_n)^2}{2\lambda_n}\right]dx_n\\ &= \frac{1}{\sqrt{2\pi\lambda_n}}\int_{-\infty}^{\infty}x_n^p\exp\left[-\frac{(2\beta\lambda_n x_n^2+x_n^2-2\,m_nx_n+m_n^2)}{2\lambda_n}\right]dx_n\\ &= \frac{1}{\sqrt{2\pi\lambda_n(2\beta\lambda_n+1)}}\int_{-\infty}^{\infty}\frac{y_n^p}{(2\beta\lambda_n+1)^{\frac{p}{2}}}\exp\left[-\frac{(y_n^2-\frac{2m_ny_n}{(2\beta\lambda_n+1)^{\frac{1}{2}}}+m_n^2)}{2\lambda_n}\right]dy_n,\end{aligned} \tag{65}$$

where:

$$y_n=(2\beta\lambda_n+1)^{\frac{1}{2}}x_n. \tag{66}$$

Completing the square gives:

$$\frac{1}{\sqrt{2\pi\lambda_n}(2\beta\lambda_n+1)^{\frac{p+1}{2}}}\int_{-\infty}^{\infty}\left\{y_n^p \exp\left[-\frac{(y_n^2-\frac{2m_n y_n}{(2\beta\lambda_n+1)^{\frac{1}{2}}}+\frac{m_n^2}{2\beta\lambda_n+1})}{2\lambda_n}\right]\right. \tag{67}$$

$$\left.\exp\left[-\frac{m_n^2}{2\lambda_n}\left(1-\frac{1}{2\beta\lambda_n+1}\right)\right]\right\}dy_n=\frac{E_0\left(y_n^p\right)}{(2\beta\lambda_n+1)^{\frac{p+1}{2}}}\exp\left[-\frac{\beta m_n^2}{(2\beta\lambda_n+1)}\right], \tag{68}$$

where the y_n are independent for all n and are distributed as

$$y_n \sim N\left(\frac{m_n}{(2\beta\lambda_n+1)^{\frac{1}{2}}},\lambda_n\right)\equiv N(m_{y_n},\lambda_n). \tag{69}$$

To evaluate $E_0\left[y_n^p\right]$, we use the binomial theorem and the fact that:

$$E_0\left[y_n-m_n\right]^k=\begin{cases}\lambda_n^{\frac{k}{2}}(k-1)(k-3)...3\cdot 1 & k \text{ even}\\ 0 & k \text{ odd.}\end{cases} \quad . \tag{70}$$

This completes the proof of Proposition 3. □

Proof of Proposition 4

For known jumps sizes, δ_j, $j=1,2,\ldots\infty$, the price at $t=0$ of a pure discount bond maturing at time $t=1$ is:

$$\begin{aligned}P_{0,1} &= a_0\exp[-r_0]+\sum_{L=1}^{\infty}\beta^L\left\{\int_0^1\int_{\tau_1}^1\cdots\int_{\tau_{L-1}}^1\left\{\exp\left[-r_0-\sum_{s=1}^{L}\delta_s(1-\tau_s\right]\right.\right. \\ &\qquad \sum_{\substack{j_1,\ldots j_L=1\\ k_1,\ldots k_L=1}}^{\infty} a_L(j,k)\prod_{m=1}^{l}\phi_{j_m}(\tau_m)\phi_{j_m}(\tau_m)\Big\}d\tau_1 d\tau_2\ldots d\tau_L\Big\} \qquad (71)\\ &= a_0\exp[-r_0]+\exp[-r_0]\sum_{L=1}^{\infty}\beta^L\sum_{\substack{j_1,\ldots j_L=1\\ k_1,\ldots k_L=1}}^{\infty} a_L(j,k) \qquad (72)\\ &\quad\times\int_0^1\int_{\tau_1}^1\cdots\int_{\tau_{L-1}}^1\prod_{m=1}^{L}\left[\exp[\delta_m\tau_m]\phi_{j_m}(\tau_m)\phi_{j_m}(\tau_m)\right]d\tau_1 d\tau_2\ldots d\tau_L. \qquad (73)\end{aligned}$$

Rearranging the order of summation and defining: $\tilde{\phi}_j(\tau_j)\equiv\exp[-\delta_j(1-\tau_j)/2]\phi_j(\tau_j)$, we obtain the expression given in Proposition 4. This completes the proof of Proposition 4. □

Proof of Proposition 5

Suppose that the jump sizes equal the contemporaneous levels of an arithmetic Brownian motion, Z_t, with drift $\tilde{\mu}$ and instantaneous standard deviation, $\tilde{\sigma}$, i.e., $\delta_j = Z_{t_j}$ for $j = 1, 2, \ldots, \infty$, and jump times, t_j, where Z_t is independent of the jump realizations and of the state variable, X_t. Given the independence of Z_t both from the jump process realization and from X_t, we can derive bond prices by initially conditioning on the jump times, evaluating the expression:

$$\mathrm{E}_0\left[\exp\left(\sum_{i=1}^{L}\delta_i(t_i-1)\right)\middle|\, t_1, t_2, t_3, \ldots\right], \tag{74}$$

and then integrating over the density of jump time density obtained in Proposition 3.

Note that, for any constant ξ and normally-distributed random variable with mean $\tilde{\mu}$ and variance $\tilde{\sigma}^2$

$$\mathrm{E}_0\left(\exp(\xi Z)\right) = \exp\left(\xi\tilde{\mu} + \frac{\xi}{2}\tilde{\sigma}^2\right). \tag{75}$$

If the jump size process is represented by the Brownian motion, Z_t, described above, then $\delta_{i+1} \equiv Z_{t_{i+1}}$ is distributed normally with mean $Z_{t_i} + \tilde{\mu}(t_{i+1} - t_i)$ and variance $\tilde{\sigma}^2(t_{i+1} - t_i)$.

Now, define $\xi \equiv t_i - 1$. Then, the expression in equation (41) is equal to:

$$\mathrm{E}_0\left[\exp\left(\sum_{i=1}^{N}\delta_i(t_i-1)\right)\middle|\, \delta_1, \delta_2 \ldots \delta_{N-1}\,;\ t_i \forall i\right] \times \exp\left(\sum_{i=1}^{N-1}\delta_i\xi_i\right) \tag{76}$$

$$\times \exp\left(\xi_N(\delta_{N-1} + \tilde{\mu}(\xi_N - \xi_{N-1})) + \frac{\xi_N^2}{2}\tilde{\sigma}^2(\xi_N - \xi_{N-1})\right), \tag{77}$$

since δ_N conditional on δ_{N-1} has mean: $\delta_{N-1} + \tilde{\mu}(\xi_N - \xi_{N-1})$ and variance $\tilde{\sigma}^2(\xi_N - \xi_{N-1})$. We now wish to evaluate the expression:

$$\mathrm{E}_0\left[\exp\left(\sum_{i=1}^{N}\delta_i(t_i-1)\right)\middle|\, \delta_1, \delta_2 \ldots \delta_{N-2}\,;\ t_i \forall i\right]. \tag{78}$$

We have

$$\exp\left(\sum_{i=1}^{N-2}\delta_i\xi_i + \xi_N\tilde{\mu}(\xi_N - \xi_{N-1})\right) \times \exp\left(\frac{\xi_N^2}{2}\tilde{\sigma}^2(\xi_N - \xi_{N-1})\right) \times \tag{79}$$

$$\exp((\xi_N + \xi_{N-1})^2(\delta_{N-2} + \tilde{\mu}(\xi_{N-1} - \xi_{N-2}))) \times \exp\left(\frac{(\xi_N + \xi_{N-1})^2}{2}\tilde{\sigma}^2(\xi_{N-1} - \xi_{N-2})\right), \tag{80}$$

and so on until we obtain the term in the second line of equation (29). This completes the proof of Proposition 5. □

Notes on the Bond Pricing Algorithm

The algorithm has two main parts. The first calculates the permutations and the corresponding moment coefficients, $a(j,k)$. The second performs the numerical integrations.

The Permutation and Moment Calculation Section

The number of permutations grows extremely rapidly as L increases. For example, if one truncates at $n = 10$ eigenfunctions, and if the parameters are consistent with an average of five jumps a year, there is significant probability weight (by which we mean probability weight exceeding 1e-5) up to the fifteenth jump, implying 1.3e+26 permutations.

Three facts are helpful in dealing with the large number of permutation. First, the order of the eigenfunctions, $\phi_{j_n}(\tau)$, $\phi_{k_n}(\tau)$, where j_n and k_n are corresponding elements of two permutations (j,k), does not matter. This reduces the number of integrations we need to perform as we may weight each $a(j,k)$ by the number of times it can occur (i.e., by 2 for each n such that $j_n \neq k_n$).

Second, the series expansion implies a relatively high rate of convergence so there are few permutations for which the corresponding $a(j,k)$ is non-negligible. We retain permutations which satisfy the following criteria. For the zero and one jump cases, we evaluate all the permutations using ten eigenfunctions. For the two jump case, we disregard any permutation for which the $a(j,k)$'s are below a prespecified tolerance level again using ten eigenfunctions. For numbers of jumps greater than two, we retain permutations that only involve the first two eigenfunctions and which satisfy the tolerance level. It is important to verify that the cumulative distribution is close to unity.

Third, for short maturity bonds, one can reach satisfactory accuracy (in that prices equal those calculated with extremely lengthy Monte Carlo) using all permutations of the first two eigenfunctions. This is particularly the case when the average frequency of jumps is low.

The Numerical Integration Section

Having selected permutations as described above, we must evaluate the corresponding numerical integrals. Dividing the interval $[0,1]$ into a series of subintervals, we employ the composite trapezoidal rule, taking the average of the function evaluated at the upper and lower bounds of each step.

To improve the accuracy of the numerical integrations, it is important to space the points in the time grid unevenly, with much finer spacing close to 0. This is because the first eigenfunction is rapidly declining in this region but is much flatter for higher values of its argument.

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Figure 1a. Key rates

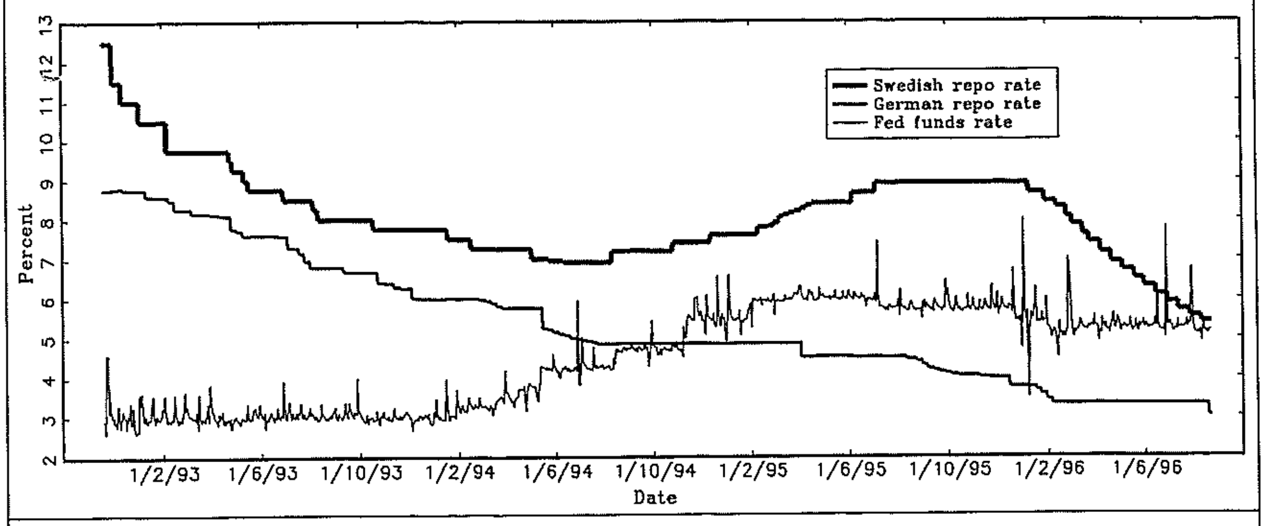


Figure 1b. Swedish Term Structure

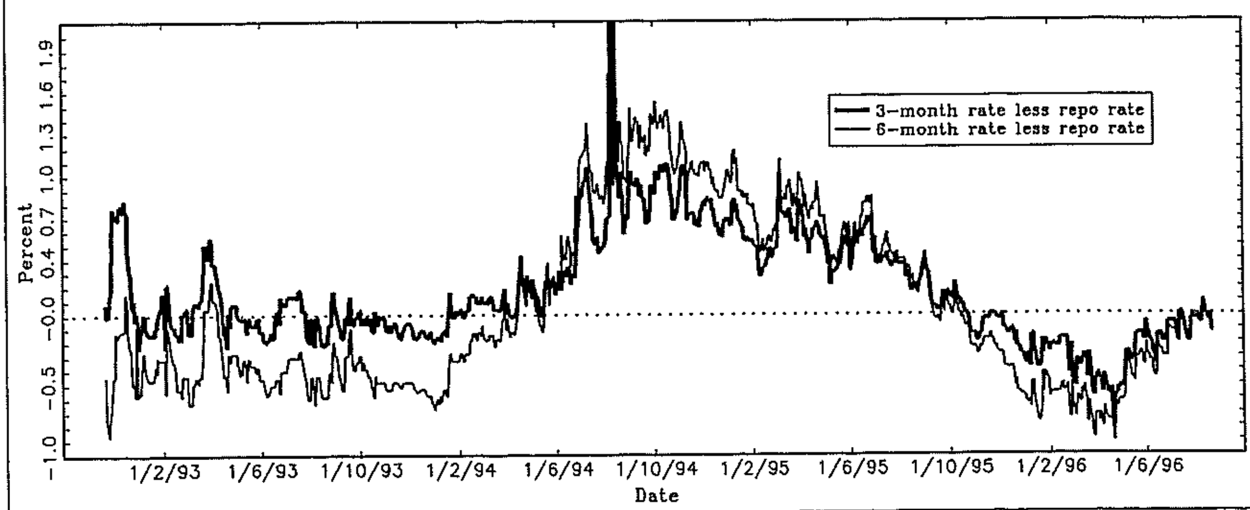


Figure 1c. German Term Structure

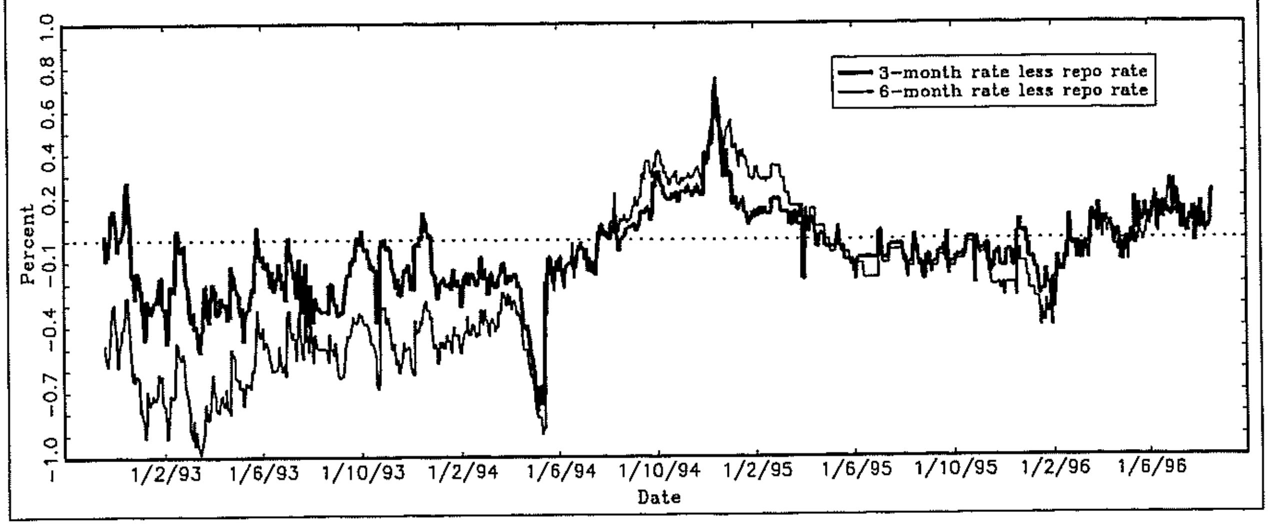


Figure 2a. Standardized 3-Month Rate Distributions

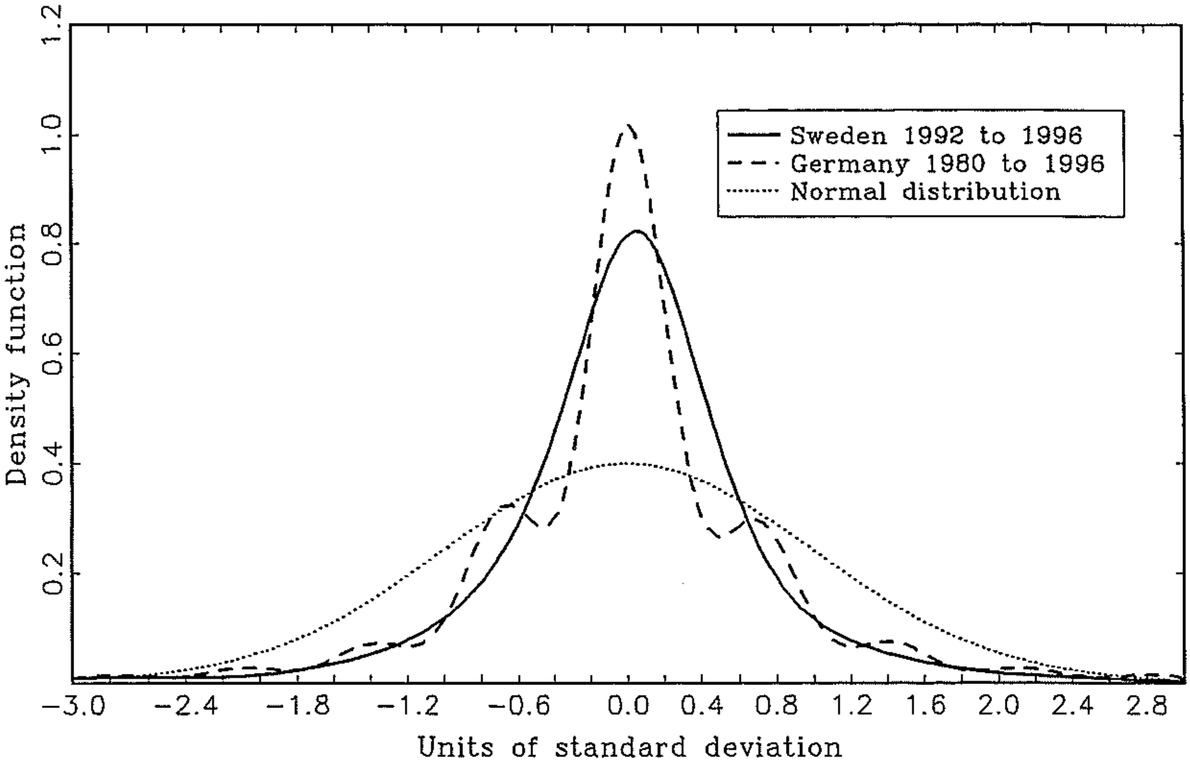


Figure 2b. Standardized US 3-Month Rate Distributions

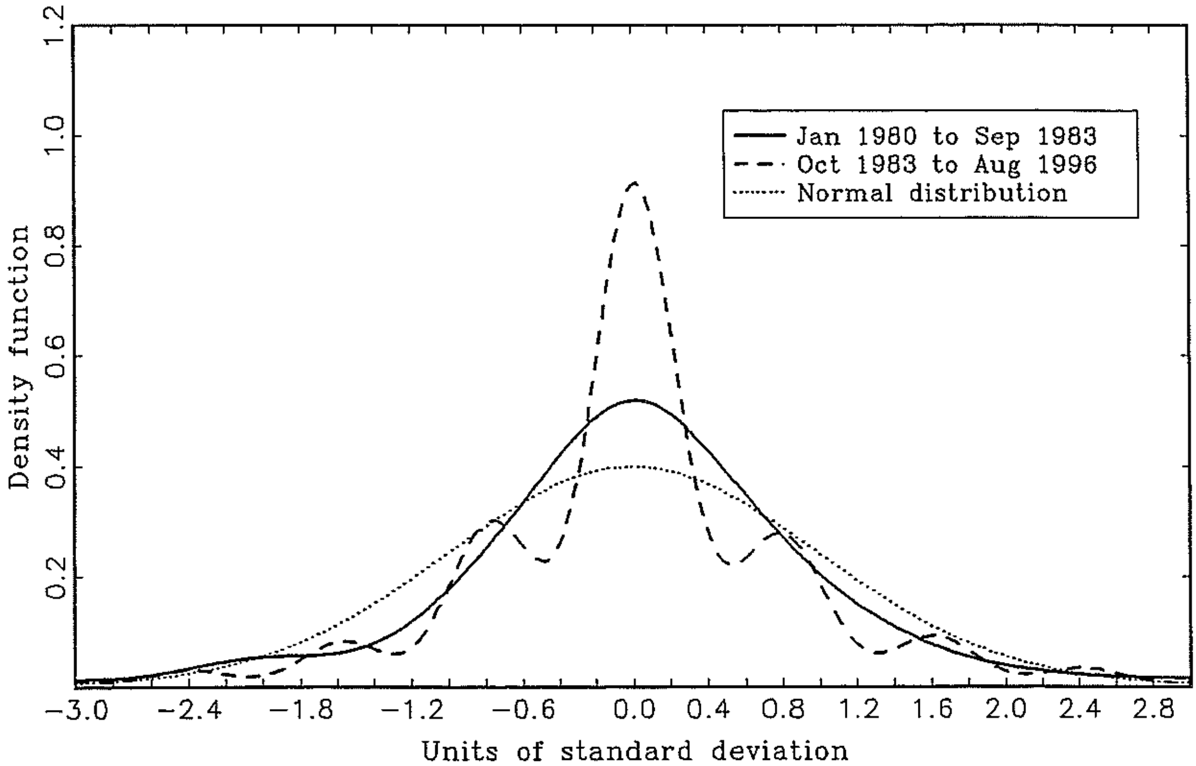


Figure 3a. Conditional Mean for N Eigenfunctions

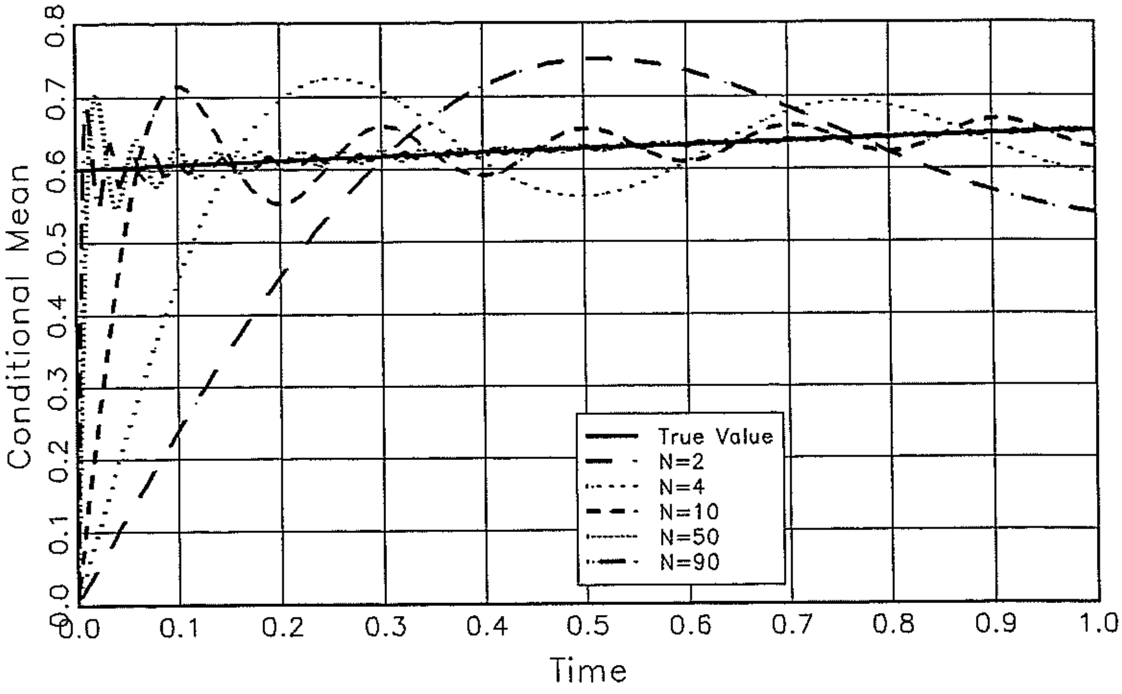


Figure 3b. Conditional Variance for N Eigenfunctions

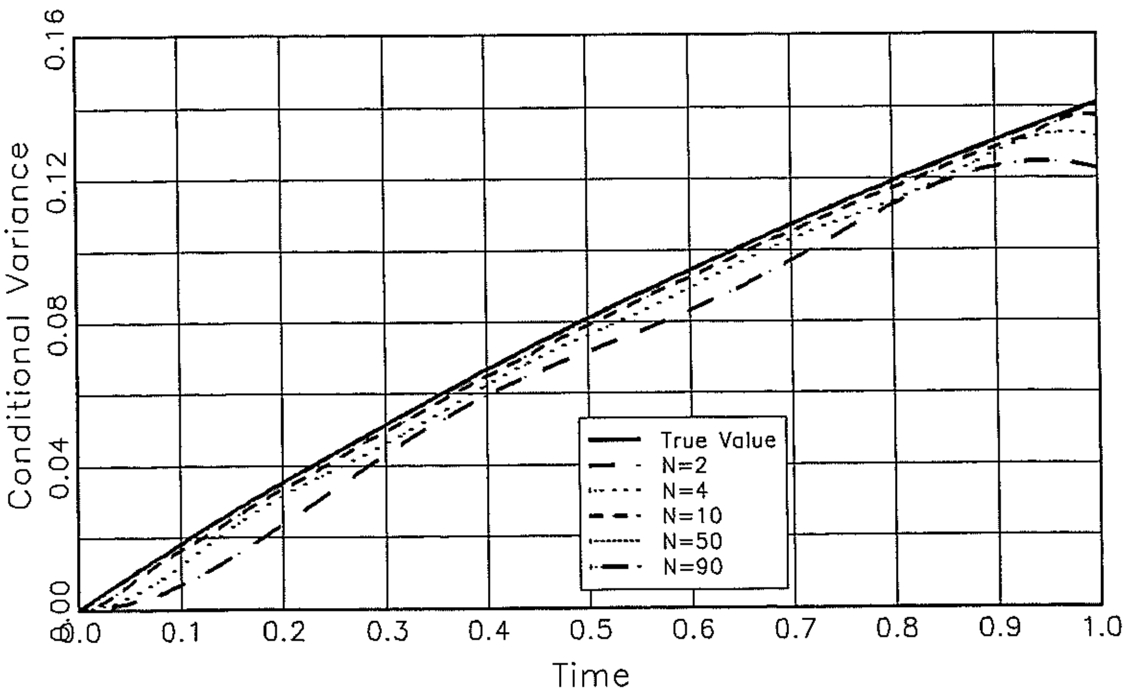
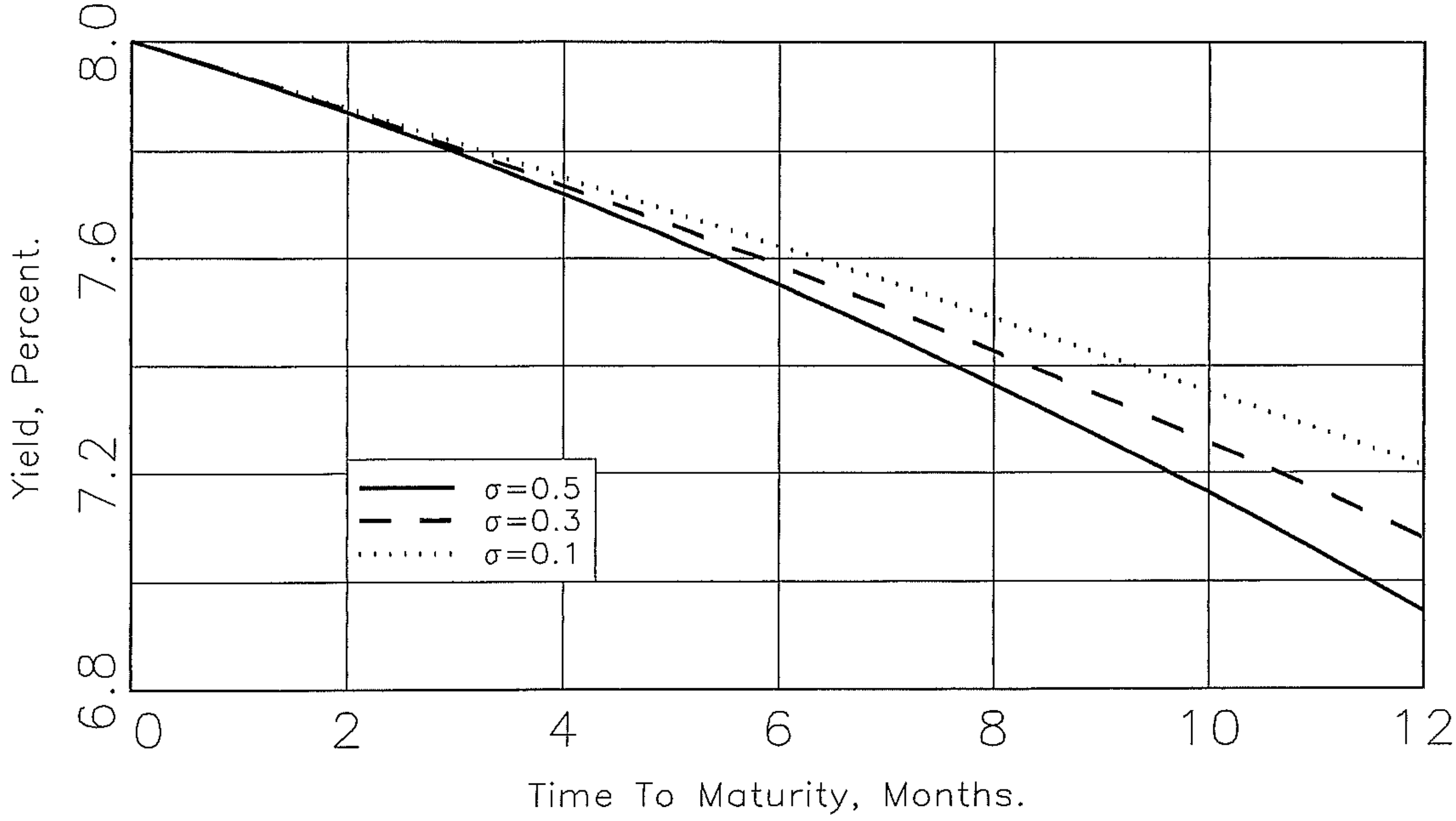
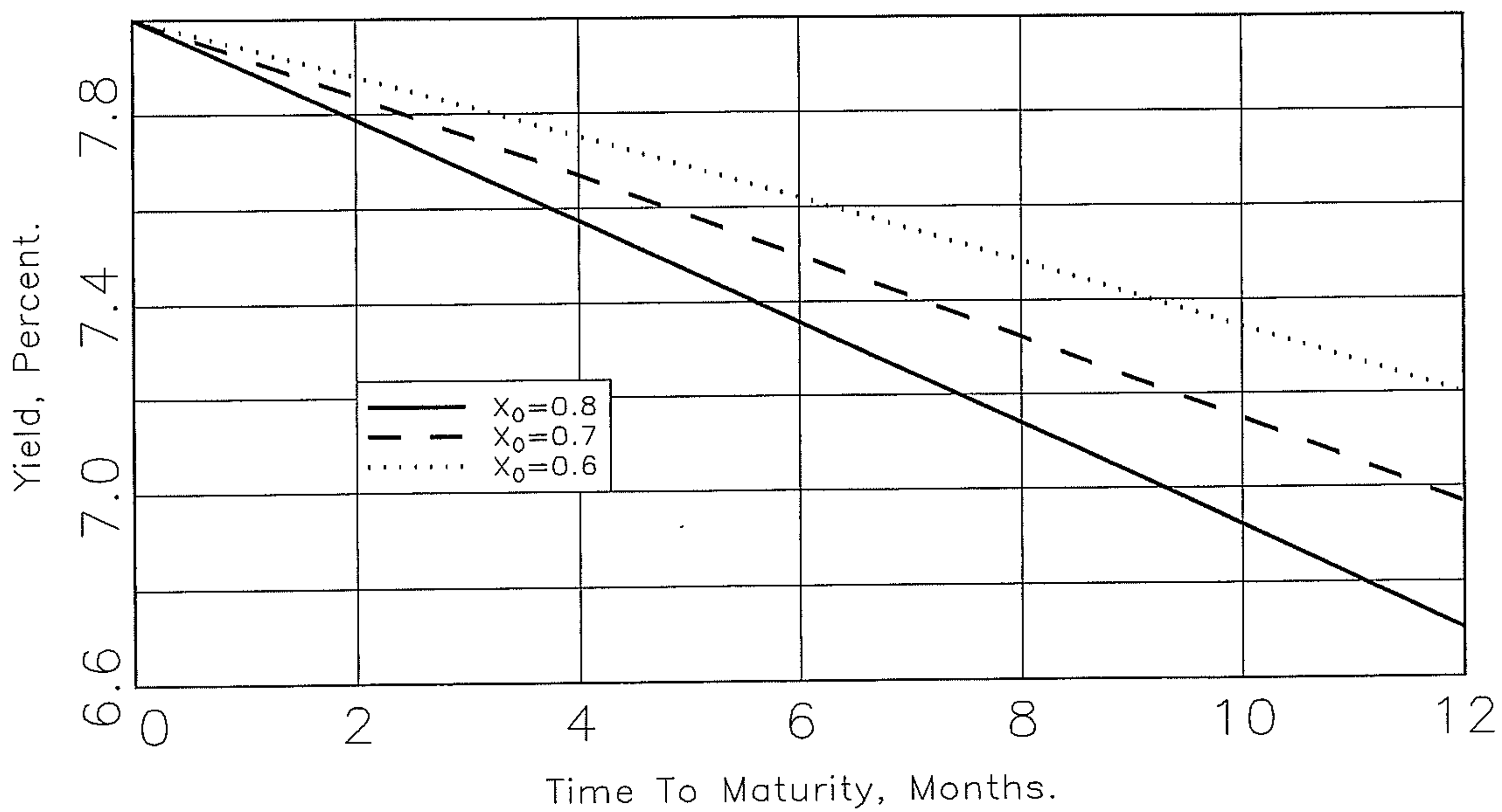


Figure 4: Yield Curves.



$\beta=8,\ \alpha=0.4,\ \Theta=0.8,\ X_0=0.6,\ \delta=-0.005.$

Figure 5: Yield Curves.



$\beta=8,\ \alpha=0.4,\ \Theta=0.7,\ \sigma=0.3,\ \delta=-0.005.$